Perturbative study of multiphoton processes in the tunneling regime

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Abstract

A perturbative study of the Schrödinger equation in a strong electromagnetic field with dipole approximation is accomplished in the Kramers-Henneberger frame. A prove that just odd harmonics appear in the spectrum for a linear polarized laser field is given, assuming that the atomic radius is much lesser than the free-electron quiver motion amplitude. Within this approximation a perturbation series is obtained in the Keldysh parameter giving a description of multiphoton processes in the tunneling regime. The theory is applied to the case of hydrogen-like atoms: The spectrum of higher order harmonics and the above-threshold ionization rate are derived. The ionization rate computed in this way determines the amplitudes of the harmonics. The wave function of the atom proves to be rigid with respect to the perturbation so that the effect of the laser field on the Coulomb potential in the computation of the probability amplitudes can be neglected as a first approximation: This approximation improves as the ratio between the amplitude of the quiver motion of the electron and the atom radius becomes larger. The semiclassical description currently adopted for harmonic generation is so rederived by solving perturbatively the Schrödinger equation.

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Availability of powerful sources of laser light has permitted, in recent years, the realization of experiments through gaseous media that have shown several new physical effects as photoionization with a number of photons absorbed by the electron well above the ionization threshold and generation of a broad range of harmonics of the laser frequency [1]. This latter effect could have a lot of technological applications and, as such, has been widely studied both theoretically and experimentally.

The possibility to turn a physical effect into a practical application is strongly linked with the availability of a satisfactory theoretical model. But, it is common belief that, due to the intensity of the laser field, no perturbation theory can be done. The main aim of this paper is then to show how perturbation theory can be straightforwardly applied also for intense laser fields and analytical expressions can be computed for any kind of multiphoton process, at least for hydrogen-like atoms. The development parameter turns out to be the the square root of the ratio between the ionization energy I_B and the ponderomotive energy U_p proportional to the intensity of the laser field, known in literature as the Keldysh parameter γ. The regime of a small Keldysh parameter characterize the so-called tunnelling regime that is the one of interest here.

Theoretical approaches to multiphoton processes are non-perturbative in nature and resort to Floquet theory as in [2], numerical methods applied directly to the Schrödinger equation as done firstly in [3] or semiclassical models [4]. On the basis of the semiclassical ideas, a quantum theory for harmonic generation has been obtained by L'Huillier and coworkers in [5]: Our theory permits to justify the main assumptions of the quantum theory of these authors, so that, in turn, the semiclassical ideas prove to be a fairly good description of harmonic generation.

The approach we apply to the Schrödinger equation for an atom in an electromagnetic field can be easily understood using a two-level model, widely used for harmonic generation [6]. This model has the Hamiltonian (here and in the following we will take $\hbar = c = 1$)

$$H = \frac{\omega_0}{2}\sigma_3 + \Omega\cos(\omega t)\sigma_1 \tag{1}$$

being ω_0 the level separation, Ω the intensity of the laser field and ω its frequency, σ_1 and σ_3 are Pauli matrices. If Ω is small with respect to ω_0, standard perturbation theory applies by interaction picture through an unitary transformation that removes the unperturbed part of the Hamiltonian: This gives a Dyson series in the small development parameter $\Omega/(\omega_0 \pm \omega)$, out of resonance. Recently, duality has been introduced in perturbation theory [7] and a dual interaction picture has been devised where one does an unitary transformation to remove the perturbation. For the above Hamiltonian one has to take $U = e^{-i\sigma_1 \frac{\Omega}{\omega_0} \sin(\omega t)}$ that yields the transformed hamiltonian [8]

$$\begin{aligned} H_F &= \frac{\omega_0}{2} e^{2i\sigma_1 \frac{\Omega}{\omega_0} \sin(\omega t)} \sigma_3 \\ &= \frac{\omega_0}{2} J_0 \left(\frac{2\Omega}{\omega}\right) \sigma_3 + \frac{\omega_0}{2} \sum_{n \neq 0} J_n \left(\frac{2\Omega}{\omega}\right) e^{in\sigma_1 \omega t} \sigma_3 \end{aligned} \tag{2}$$

where now, perturbation theory can be done for $\Omega \gg \omega_0, \omega$. We see straightforwardly that the unperturbed part of the Hamiltonian is "dressed" by the laser field and so, the energy levels are shifted. Then, the perturbation has odd and even harmonics of the laser frequency and both can appear in the spectrum. But, probability amplitudes that enters in the computation of the spectrum do not depend on the unitary transformations one does on the Hamiltonian and the states. So, we have sketched the physics of the two-level model in an intense monochromatic field just through dual interaction picture. Although, as we will show, the two-level model does not apply for current experiments with atomic samples as in this case one observes just odd harmonics in agreement with our full theory and it is not just a problem of a proper experimental setup, nevertheless it could have a wide range of applications in magnetic resonance experiments, for some other kind of media as optical cavities [9] or wherever the conditions one meets for atomic samples are no more fulfilled.

The dual interaction picture applies in the same way also to the Schrödinger equation in a semiclassical laser field and in the dipole approximation, as currently treated in literature [5]. The correspondence with the two-level model above is remarkable. The Hamiltonian in this case is

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) + \frac{e}{m}\mathbf{A}(t)\mathbf{p} + \frac{e^2}{2m}\mathbf{A}^2(t). \tag{3}$$

By the unitary transformation $U(t) = \exp\left(-i\frac{e}{m}\int_0^t dt'\mathbf{A}(t')\mathbf{p} - i\frac{e^2}{2m}\int_0^t dt'\mathbf{A}^2(t')\right)$

the above Hamiltonian transforms into

$$H_{KH} = U^\dagger(t)\left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\right)U(t) = \frac{\mathbf{p}^2}{2m} + V[\mathbf{x} + \mathbf{a}(t)] \tag{4}$$

being $\mathbf{a}(t) = -\frac{e}{m}\int_0^t dt'\mathbf{A}(t')$. This is the well-known Kramers-Henneberger Hamiltonian and the unitary transformation above define the so called Kramers-Henneberger frame [10] that shows as the effect of the electromagnetic field is to introduce a time-dependent translation on the potential of the unperturbed Hamiltonian by a length $\mathbf{a}(t)$. The laser field can be modeled as $\mathbf{A}(t) = -\frac{E(t)}{\omega\sqrt{1+\xi^2}}[\hat{\mathbf{x}}\cos(\omega t) + \xi\hat{\mathbf{y}}\sin(\omega t)]$ for a general ellipticity parameter ξ. Here we consider the simplest case of a linear polarization $\xi = 0$ and an instant rising of the laser field, that is $E(t) = const$. So, one has [1]

$$H_{KH} = \frac{\mathbf{p}^2}{2m} + \int_{-\lambda_L}^{\lambda_L}\frac{dx'}{\pi}\frac{V(x-x',y,z)}{\sqrt{\lambda_L^2 - x'^2}} + \sum_{k=1}^{+\infty} i^k[e^{ik\omega t} + (-1)^k e^{-ik\omega t}]v_k(\mathbf{x}) \tag{5}$$

with

$$v_k(\mathbf{x}) = \int_{-\lambda_L}^{\lambda_L}\frac{dx'}{\pi}V(x-x',y,z)\frac{T_k\left(\frac{x'}{\lambda_L}\right)}{\sqrt{\lambda_L^2 - x'^2}} \tag{6}$$

being $T_k(x) = \cos(k\arccos(x))$ the k-th Chebyshev polynomial of first kind and $\lambda_L = \frac{eE}{m\omega^2} = \sqrt{\frac{4U_p}{m}}\frac{1}{\omega}$ the maximum free-electron quiver motion excursion. This length is pivotal in the study of atoms in an intense laser field as generally one has $\lambda_L \gg a$, being $a = \frac{1}{mZe^2}$ the Bohr radius. One can see that, as for the two-level model, we have the potential of the unperturbed part of the Hamiltonian “dressed” by the laser field and all the harmonics, odd and even, are present in the perturbation. We can now show that, in all the current experiments where the potential $V(\mathbf{x})$ depends just on $r = |\mathbf{x}|$ and $\lambda_L \gg a$, being a the Bohr radius of the atoms in the sample, then just odd harmonics appear in the spectrum. Indeed, we can rewrite eq.(6) as

$$v_k(\mathbf{x}) = \int_{-1}^{1} dx' V(\sqrt{(x-\lambda_L x')^2 + y^2 + z^2})\frac{T_k(x')}{\pi\sqrt{1-x'^2}}. \tag{7}$$

If the laser field is enough intense, a series in $\frac{a}{\lambda_L}$ is obtained if one develops eq.(7) in Taylor series as

$$v_k(\mathbf{x}) = \int_{-1}^{1} dx'\, V(\sqrt{(x-\lambda_L x')^2 + y^2 + z^2})\Big|_{x=0,y=0,z=0}\frac{T_k(x')}{\pi\sqrt{1-x'^2}}$$

$$-x\int_{-1}^{1}dx'V'(\lambda_L|x'|)\frac{x'}{|x'|}\frac{T_k(x')}{\pi\sqrt{1-x'^2}}+\cdots. \tag{8}$$

Despite its appearance, the terms of this series can be evaluated for a Coulomb potential and proved to be finite assuring the convergence. This is due to the fact that in this case the integrals can be computed analitically. Then, from the above expression two main conclusions can be drawn. Firstly, multiphoton effects are due to a dipole induced on the atom in the same direction as the electric field of the laser and secondly, Chebyshev polynomials have a definite parity and due to the symmetrical range of integration, only odd polynomials give a non-null contribution to the second term, while the first term has no physical consequences and in the following will be neglected. So, only odd harmonics contribute to the spectrum while, even harmonics are quadrupole radiation and then strongly depressed. Indeed, for a Coulomb potential one obtains

$$v_{2n+1}(\mathbf{x})\approx -i(-1)^n\frac{x}{\lambda_L}(2n+1)\frac{Ze^2}{\lambda_L}. \tag{9}$$

This result, that does not involve any other approximation beside the simmetry of the potential and the amplitude of the quiver motion of the electron with respect to the atomic radius, supports in some way the physical view recently given in [11], where it is assumed that the electron recolliding with the atomic core, emits bremsstrahlung radiation that is cut off at the maximum amplitude of the quiver motion of the electron, producing in this way just odd harmonics.

To complete the above discussion before introducing perturbation theory, we have to study the "dressed" potential v_0. This should be managed differently from the time-dependent part. Indeed, we have to separate the original potential $V(r)$ from the shifts induced by the laser field on the energy levels of the atom. This can be obtained by a Taylor expansion as

$$\begin{aligned}
v_0(\mathbf{x}) &= \int_{-1}^{1}\frac{dx'}{\pi}\frac{V(\sqrt{(x-\lambda_L x')^2+y^2+z^2})}{\sqrt{1-x'^2}} \\
&= V(r)+\delta_L V(\mathbf{x}) \\
&= V(r)+\frac{\lambda_L^2}{4r^3}\left[V'(r)y^2+V'(r)z^2+V''(r)x^2r\right]+\cdots
\end{aligned} \tag{10}$$

where is seen that only even terms survive and higher order terms fall off very rapidly with r. The above expression assumes a very simple form for a Coulomb potential

$$v_0(\mathbf{x}) = -\frac{Ze^2}{r}\left[1+\sum_{n=1}^{+\infty}A_n\left(\frac{\lambda_L}{r}\right)^{2n}P_{2n}\left(\frac{x}{r}\right)\right] \qquad (11)$$

being $A_n = \int_{-1}^{1} dx x^{2n}/(\pi\sqrt{1-x^2})$ and P_n the n-th Legendre polynomial. This way to express the dressed Coulomb potential gives us a way to prove that the wave function is "rigid" with respect to the perturbation using standard Rayleigh-Schrödinger perturbation scheme, for the kind of problems we discuss here. But, it should be pointed out that for stabilization things are quite different [12].

The equations for the amplitudes are given by

$$\begin{aligned} i\dot{a}_m(t) &= \sum_{n\neq m} a_n(t)\langle m|\delta_L V(\mathbf{x})|n\rangle e^{-i(\tilde{E}_n-\tilde{E}_m)t} + \\ & \sum_n\sum_{k=1}^{+\infty} i^k a_n(t)\langle m|v_k(\mathbf{x})|n\rangle[e^{-i(\tilde{E}_n-\tilde{E}_m-k\omega)t}+(-1)^k e^{-i(\tilde{E}_n-\tilde{E}_m+k\omega)t}] \end{aligned} \qquad (12)$$

having set $\tilde{E}_n = E_n + \langle n|\delta_L V(\mathbf{x})|n\rangle$, being $\delta_L V(\mathbf{x})$ the part of the static potential due to the laser field. At this point, all the machinery of standard perturbation theory applies [13]. For our aim, we have to show that the Rayleigh-Schrödinger part gives indeed a small contribution to the amplitudes. By assuming the atom initially in its ground state, this contribution is

$$a_m^{RS}(t) \approx \frac{\langle m|\delta_L V(\mathbf{x})|1\rangle}{\tilde{E}_1-\tilde{E}_m}(e^{-i(\tilde{E}_1-\tilde{E}_m)t}-1). \qquad (13)$$

Using eq.(11) is easy to verify that no contribution comes for $m=2$ as $\langle 2|\delta_L V(\mathbf{x})|1\rangle = 0$ but the degeneracy of level 2 is removed by the dressed potential as one has $\langle m=2, l=1, l_z=0|\delta_L V(\mathbf{x})|m=2, l=1, l_z=0\rangle = Ze^2/(240a)(\lambda_L/a)^2$ and $\langle m=2, l=1, l_z=\pm1|\delta_L V(\mathbf{x})|m=2, l=1, l_z=\pm1\rangle = -Ze^2/(480a)(\lambda_L/a)^2$ while $\langle m=2, l=0, l_z=0|\delta_L V(\mathbf{x})|m=2, l=0, l_z=0\rangle = 0$. Indeed, one can see that all the states having m even do not give a first order contribution even if the level shift is not null, while the level-shift is always 0 when $l=0$. Instead, for $m=3$ one has e.g. $\langle m=3, l=2, l_z=0|\delta_L V(\mathbf{x})|m=1, l=0, l_z=0\rangle = Ze^2\sqrt{150}/(10800a)(\lambda_L/a)^2$ and for the level shifts $\langle m=3, l=2, l_z=0|\delta_L V(\mathbf{x})|m=3, l=2, l_z=0\rangle =$

$Ze^2/(5670a)(\lambda_L/a)^2 - Ze^2/(136080a)(\lambda_L/a)^4$ and $\langle 1|\delta_L V(\mathbf{x})|1\rangle = 0$, so the correction of eq.(13) turns out to be

$$a^{RS}_{3,2,0}(t) \approx -\frac{\frac{\sqrt{150}}{10800}\left(\frac{\lambda_L}{a}\right)^2}{\frac{4}{9}+\frac{1}{5760}\left(\frac{\lambda_L}{a}\right)^2-\frac{1}{136080}\left(\frac{\lambda_L}{a}\right)^4}(e^{i\frac{8}{9}E_1 t}-1) \tag{14}$$

that is indeed negligeable and the wave function turns out to be "rigid" with respect to the deformations introduced by the laser field. This is even more true as larger become the ratio λ_L/a. The reason for this is that only a finite number of terms of eq.(11) give a non-null contribution to the matrix elements. It is interesting to note that for stabilization of an atom in intense laser field the situation is exactly the contrary as one should be able to diagonalize the Hamiltonian $H_0 = \mathbf{p}^2/2m + v_0(\mathbf{x})$ being the time-dependent part negligible, an approximation that becomes exact in the limit of infinite frequency of the laser field [1, 12].

Then, the iterative procedure to solve eq.(12) can be applied to compute the probability transition for any process. This approach implies that off-resonant contributions should be sistematically neglected. In this way, a golden rule is straightforwardly obtained as

$$P_{i\rightarrow f} = 2\pi\sum_{n=1}^{+\infty}|\langle i|v_n(\mathbf{x})|f\rangle|^2\delta\left[\tilde{E}_f-\tilde{E}_i-n\omega\right] \tag{15}$$

from which several results for multiphoton processes can be obtained. It is assumed a continuum of final states to sum over so that excited levels can decay, otherwise quantum resonance theory applies [15] and Rabi flopping is obtained. In any case, going to second order gives a.c.Stark shifts of the energy levels. Rabi frequency due to resonance with the k-th harmonic of the perturbation with two levels m and n of the atom is (ref.[15]) $\frac{\Omega_R}{2} = |\langle m|v_k(\mathbf{x})|n\rangle|$.

From eq.(15) we can easily compute the rate of above threshold ionization. For hydrogen-like atoms [eq.(9)] and assuming the atom initially in its ground state one has

$$\Gamma = \frac{32}{3}\frac{\omega^2}{U_p}\gamma^2\sum_{n=n_0}^{+\infty}\left[\frac{I_B}{(2n+1)\omega}\right]^{\frac{5}{2}}\left[1-\frac{I_B}{(2n+1)\omega}\right]^{\frac{3}{2}} \tag{16}$$

being n_0 the minimun integer for which $(2n_0+1)\omega - I_B \geq 0$. It has been used the fact that, as shown above, for the ground state of hydrogen-like atoms there is no shift by the part of the static potential due to the laser field, that is $\langle 1|\delta_L V(\mathbf{x})|1\rangle = 0$ for Coulomb potential. Beside, a plane wave is assumed for the particle in the final state to make computation simpler. By taking ref. [14] for experimental results , we can check the above expression for helium and neon that show a large plateau in the tunneling regime. So, we have $U_p = 155$ eV being the intensity 1.5×10^{15} W/cm^2, $\omega = 1.177$ eV and $I_B = 24.59$ eV. Then, $\gamma \approx .4$ and $\Gamma \approx 0.026$ eV, that is small as it should be expected. The same computation for neon gives approximatively 0.02 eV.

To analyse the question of harmonic generation, one has to compute $< x >= \langle\Psi(t)|x|\Psi(t)\rangle$. To complete this computation, we assume that no intermediate resonance is present and will justify this assumption a posteriori through the quantum resonance theory of ref.[15], that here applies. So, let us take an atom initially prepared in its ground state as to have $a_i(0) = \delta_{i0}$. From eq.(12) one has

$$\begin{aligned} a_m(t) &= \delta_{m0} + \frac{\langle m|\delta_L V(\mathbf{x})|0\rangle}{\tilde{E}_0 - \tilde{E}_m - i\epsilon} e^{-i(\tilde{E}_0 - \tilde{E}_m - i\epsilon)t} + \\ & \sum_{k=1}^{+\infty} i^k a_n(t)\langle m|v_k(\mathbf{x})|0\rangle \left[\frac{e^{-i(\tilde{E}_0-\tilde{E}_m-k\omega-i\epsilon)t}}{\tilde{E}_0 - \tilde{E}_m - k\omega - i\epsilon} + (-1)^k \frac{e^{-i(\tilde{E}_0-\tilde{E}_m+k\omega-i\epsilon)t}}{\tilde{E}_0 - \tilde{E}_m + k\omega - i\epsilon}\right] + \end{aligned} \tag{17}$$

with the limit $\epsilon \to 0$ understood as to have $\frac{1}{x \pm i0} = P\frac{1}{x} \mp i\pi\delta(x)$, being P the principal value. As is customary in perturbation theory, we keep just those terms that are near resonant with the harmonics of the perturbation: The only possibility left is the continuous spectrum, as it should be with the current understanding of harmonic generation. So, we take

$$a_{\mathbf{p}}(t) \approx -\sum_{k=1}^{+\infty} i^k \langle \mathbf{p}|v_k(\mathbf{x})|0\rangle(-1)^k \frac{e^{i(E_{\mathbf{p}}-\tilde{E}_0-k\omega+i\epsilon)t}}{E_{\mathbf{p}} - \tilde{E}_0 - k\omega + i\epsilon} \tag{18}$$

being $\mathbf{p}$ the momentum of the particle in the continuous part of the spectrum. Now, we specialise this expression to the case of hydrogen-like atoms having

$$a_{\mathbf{p}}(t) \approx \frac{Ze^2}{\lambda_L^2} \sum_{n=0}^{+\infty} (2n+1)\langle \mathbf{p}|x|0\rangle \frac{e^{i(E_{\mathbf{p}}-E_0-(2n+1)\omega+i\epsilon)t}}{E_{\mathbf{p}} - E_0 - (2n+1)\omega + i\epsilon}. \tag{19}$$

Then, for the dipole moment one has

$$< x > \approx \sum_{\mathbf{p}} a_{\mathbf{p}}(t) e^{-i(E_{\mathbf{p}}-E_0)t} \langle 0|x|\mathbf{p}\rangle + c.c. \tag{20}$$

After passing from the sum to integration through $\sum_{\mathbf{p}} \rightarrow V \int \frac{d^3p}{(2\pi)^3}$ and taking for the final state a plane wave, one gets the final expression for the harmonic spectrum

$$< x > \approx -\frac{64}{3^{\frac{9}{2}}}\frac{Ze^2\omega}{U_p^2}\gamma^5 \sum_{n=n_0}^{+\infty} \frac{x_n^{\frac{3}{2}}}{\left(x_n + \frac{\gamma^2}{3}\right)^5} \sin((2n+1)\omega t) \tag{21}$$

being $x_n = \frac{(2n+1)\omega - I_B}{3U_p}$. The normalization to $3U_p$ for x_n originates from the fact that, from the above expression, the intensities of the harmonics reduce as the factor $3U_p$ increases. Then, if the Keldysh parameter γ is enough small we can take

$$< x > \approx -\frac{64}{3^{\frac{9}{2}}}\frac{Ze^2\omega}{U_p^2}\gamma^5 \sum_{n=n_0}^{+\infty} \frac{1}{x_n^{\frac{7}{2}}} \sin((2n+1)\omega t) \tag{22}$$

so that, only for $x_n \leq 1$ the harmonic amplitudes are large. This is the approximate cut-off law found out through semiclassical methods in ref.[4]. It should also be stressed the existence of a minimun harmonic order n_0 that should be expected due to the close connection between harmonic generation and multiphoton ionization. Indeed, this lower bound comes out from the phase space through the integration of the Dirac function both for the golden rule (15) and for the computation of the dipole moment $< x >$. Then, one gets $n_0 = 10$ and 9 for helium and neon respectively, that means harmonic 21 for the starting point of the spectrum in the regime of interest. It should be pointed out that the above equation for $< x >$ has to take properly into account the ionization rate Γ of eq.(16) as to have at last

$$< x > \approx -\frac{64}{3^{\frac{9}{2}}}\frac{Ze^2\omega}{U_p^2}\gamma^5 \sum_{n=n_0}^{+\infty} \frac{x_n^{\frac{3}{2}}}{\left(x_n + \frac{\gamma^2}{3}\right)^5} \sin((2n+1)\omega t) e^{-\Gamma t}. \tag{23}$$

One can estimate the constant factor that determines the amplitude of the harmonics $\frac{64}{3^{\frac{9}{2}}}\frac{Ze^2\omega}{U_p^2}\gamma^5$. Indeed, for helium one obtains approximately $.32 \times 10^{-8}$ eV^{-1} and for neon about $.12 \times 10^{-7}$ eV^{-1}, showing, as it should be, a larger amplitude for neon.

A further analysis concerns the effect of intermediate resonances on the spectrum of harmonics. On the basis of the theory of ref.[15], one can write down eq.(18) as

$$a_{\mathbf{p}}(t) \approx -\sum_{k=1}^{+\infty} i^k \langle \mathbf{p}|v_k(r)|0\rangle (-1)^k \frac{e^{i(E_{\mathbf{p}}-\tilde{E}_0-k\omega+i\epsilon)t}}{E_{\mathbf{p}}-\tilde{E}_0-k\omega+i\epsilon} \cos\left(\frac{\Omega_R}{2}t\right) \quad (24)$$

with Ω_R the Rabi frequency computed taking in account the resonances between the ground state and other discrete levels. To compute the above expression we assumed that the atom is initially prepared in its ground state so that, $a_0(t) = \cos\left(\frac{\Omega_R}{2}t\right)$, essentially the rotating wave approximation. It is easy to realize that one gets the harmonics in the spectrum shifted by the quantity $\pm\frac{\Omega_R}{2}$.

The theory above could have wide applicability as, in principle for any multiphoton process one is able to compute analytical formulae to compare with experimental results. For instance, an improvement easy to implement is to use a full Coulomb wave function also for the final state in the above computations. On the other hand, even if major features of multiphoton processes are described by this theory, several problems are surely opened up as the applicability of the theory for an ellipticity parameter $\xi \neq 0$, the introduction of a slower rising of the laser field or how to take into account all the features that real experiments have for harmonic generation. Beside, when the intensity of the laser field becomes too high the above approach should be properly modified as relativistic effects enter into the physical picture and, e.g. even harmonics can also be significant [16]. Experiments to generate even harmonics are also carried out using solid surfaces as in [17]. Anyhow, it should be stressed how the possibility to derive a perturbative solution to the Schrödinger equation could give a chance to check models of multiphoton physics that no other approach offers.

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